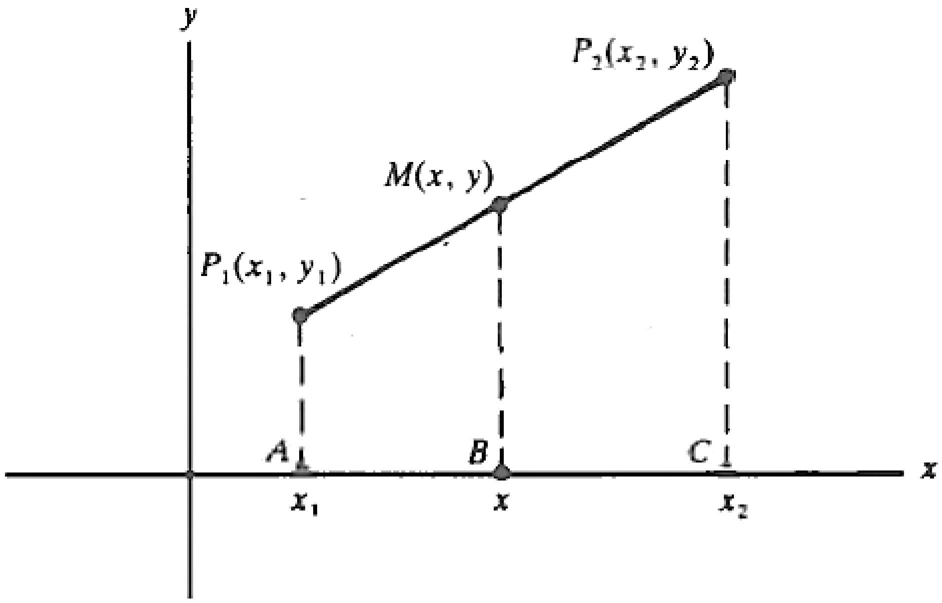
7.2.2 The Midpoint Formulas

The point $M(x, y)$ that is the midpoint of the segment connecting the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ has the coordinates

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$

Thus, the coordinates of the midpoints are the averages of the coordinates of the endpoints. See Fig. 2-6.



To see this, let A, B, C be the projections of P_1 M, P_2 on the *x* axis. The *x* coordinates of A, B, C are x_1 x, x_2. Since the lines P_1A, MB, and P_2C are parallel, the ratios $\overline{P_1M}/\overline{MP_2}$ and$\overline{AB}/\overline{BC}$ are equal. Since $\overline{P_1M} = \overline{MP_2}$, $\overline{AB} = \overline{BC}$. Since $\overline{AB} = x - x_1$ and $\overline{BC} = x_2 - x$

$$x - x_1 = x_2 - x$$

$$2x = x_2 + x_1$$

$$x = \frac{x_1 + x_2}{2}$$

(The same equation holds when P_2 is to the left of P_1, in which case $\overline{AB} = x - x_1$ and $\overline{BC} = x_2 - x$). Similarly, $y = \frac{y_1 + y_2}{2}$

EXAMPLES:

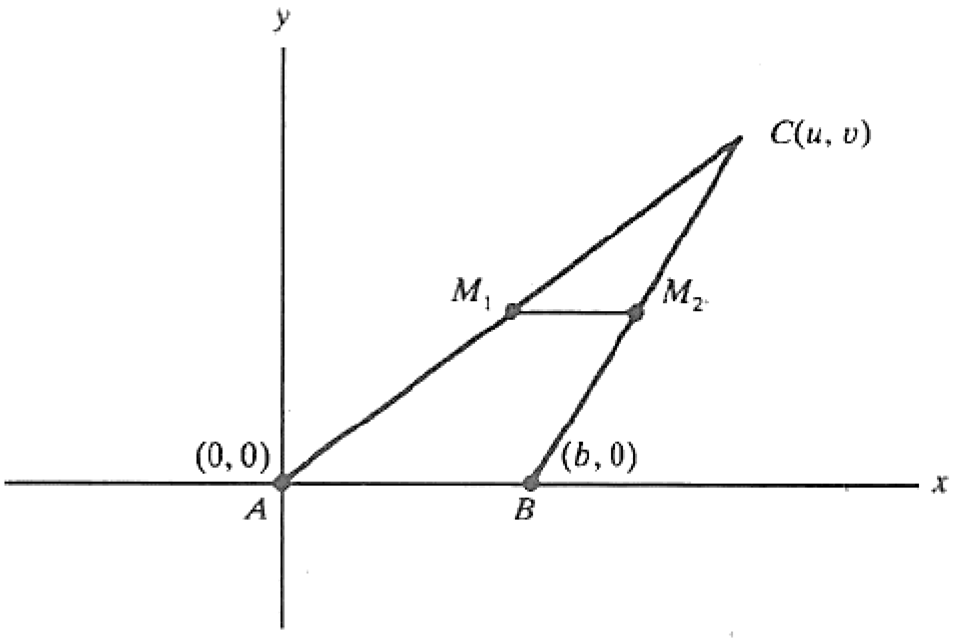
(a) The midpoint of the segment connecting (2, 9) and (4, 3) is

$$\left(\frac{2 + 4}{2}, \frac{9 + 3}{2}\right) = (3{,}6)$$

(b) The point halfway between (-5, 1) and (1, 4) is

$\left(\frac{-5+1}{2}, \frac{1+4}{2}\right) = \left(-2, \frac{5}{2}\right).$

EXAMPLE 2: Let us prove analytically that the segment joining the midpoints of two sides of a triangle is one-half the length of the third side. Construct a coordinate system so that the third side AB lies on the positive x axis, A is the origin, and the third vertex C lies above the x axis, as in Fig. 2-7.



Let ***b*** be the ***x*** coordinate of ***B***. (In other words, let $b = \overline{AB}$.) Let **C** have coordinates (u, v). Let M_1 and M_2, be the midpoints of sides AC and BC, respectively. By the midpoint formulas (2.2), the coordinates of M_1 are $\left(\frac{u}{2}, \frac{v}{2}\right)$ and the coordinates of M_2 are $\left(\frac{u+b}{2}, \frac{v}{2}\right)$. By the distance formula (2.1),

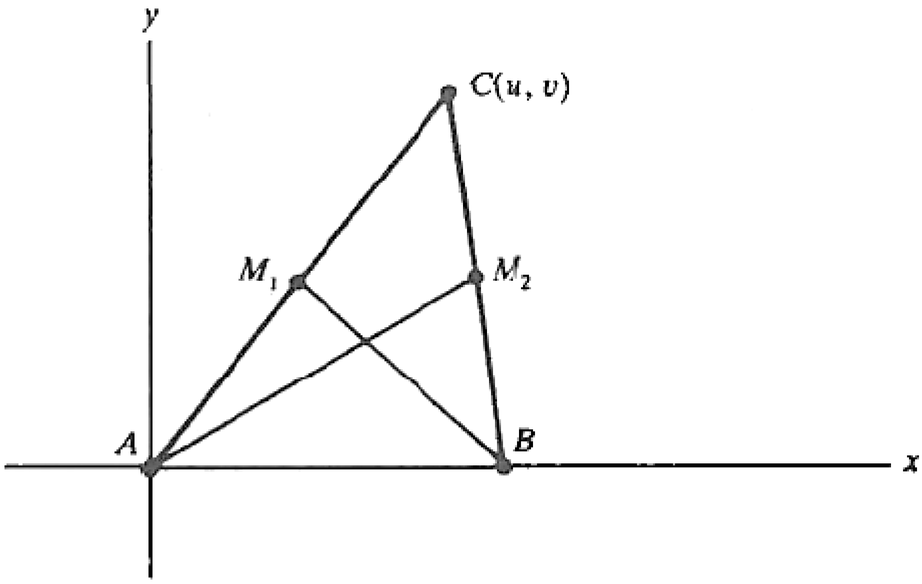
$$\overline{M_1M_2} = \sqrt{\left(\frac{u}{2} - \frac{u+b}{2}\right)^2 + \left(\frac{v}{2} - \frac{v}{2}\right)^2} = \sqrt{\left(\frac{b}{2}\right)^2} = \frac{b}{2}$$

which is half the length of side AB.

Example 3: Prove analytically that, if the medians to two sides of a triangle are equal, then those sides are equal. (Recall that a ***median*** of a triangle is a line segment joining a vertex to the midpoint of the opposite side.)

In ΔABC, let M_1 and M_2 be the midpoints of sides AC and BC, respectively. Construct a coordinate system so that A is the origin, B lies on the positive x axis, and C lies above the x axis (see Fig. 2-8). Assume that $\overline{AM_2} = \overline{BM_1}$. We must prove that $\overline{AC} = \overline{BC}$. Let b be the x coordinate of B, and let C have coordinates (u, v). Then, by the midpoint formulas, M_x has coordinates $\left(\frac{u}{2}, \frac{v}{2}\right)$ and M, has coordinates $\left(\frac{u+b}{2}, \frac{v}{2}\right)$

Hence,



$$\overline{AM_2} = \sqrt{\left(\frac{u+b}{2}\right)^2 + \left(\frac{v}{2}\right)^2}$$

And

$$\overline{BM_1} = \sqrt{\left(\frac{u}{2} - b\right)^2 + \left(\frac{v}{2}\right)^2}$$

Since $\overline{AM_2} = \overline{BM_1}$,

$$\left(\frac{u+b}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = \left(\frac{u}{2} - b\right)^2 + \left(\frac{v}{2}\right)^2 = \left(\frac{u+2b}{2}\right)^2 + \left(\frac{v}{2}\right)^2$$

Hence, $\frac{(u+b)^2}{4} + \frac{v^2}{4} = \frac{(u+2b)^2}{4} + \frac{v^2}{4}$ and therefore $(u+b)^2 = (u+2b)^2$. So $u + b = \pm(u + 2b)$. If $u + b = u + 2b$, then $b = -2b$ and therefore, $b = 0$, which is impossible, since $A \neq B$. Hence, $u + b = -(u - 2b) = -u + 2b$, whence $2u = b$. Now

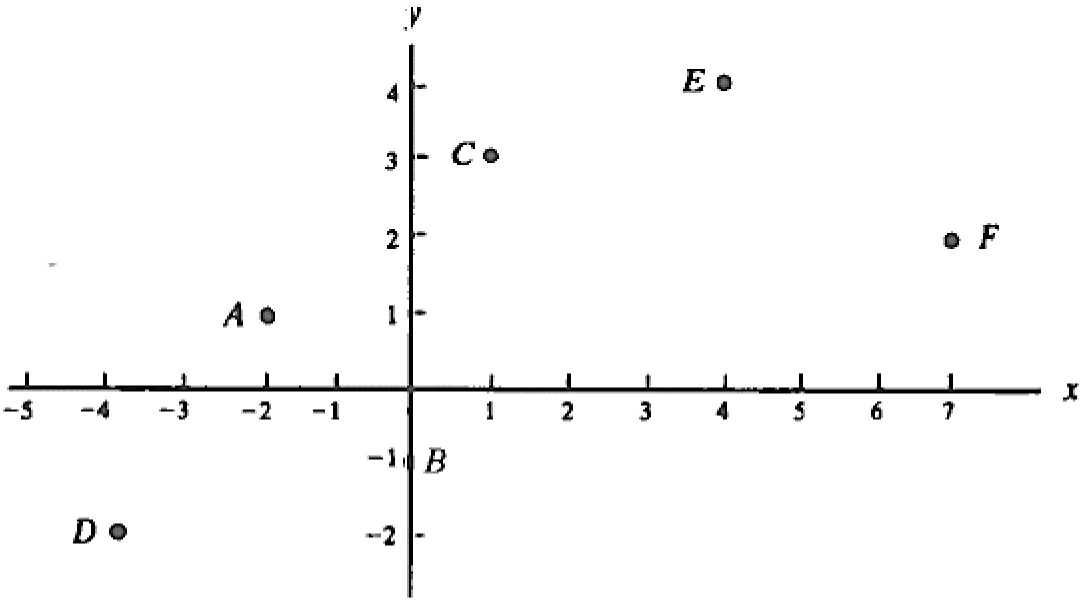
$$\overline{BC} = \sqrt{(u-b)^2 + (v)^2} = \sqrt{(u-2b)^2 + (v)^2} = \sqrt{(-u)^2 + v^2} = \sqrt{u^2 + v^2}$$

And $\overline{AC} = \sqrt{u^2 + v^2}$

Thus, $\overline{AC} = \overline{BC}$

SELF-ASSESSMENT ACTIVITY

1. Show that the distance between a point P(x, y) and the origin is $\sqrt{x^2 + y^2}$.
2. Is the triangle with vertices A(l, 5), B(4, 2), and C(5, 6) isosceles?
3. Is the triangle with vertices A(-5, 6), 5(2, 3), and C(5, 10) a right triangle?
4. Find the coordinates (x, y) of the point Q on the line segment joining P_1(l,2) and P_2(6,7), such that Q divides the segment in the ratio 2:3, that is, such that $\frac{\overline{P_1 Q}}{\overline{QP_2}} = \frac{2}{3}$
5. In the Figure below, find the coordinates of points A, B, C, D, E, and F.



6. Draw a coordinate system and show the points having the following coordinates: (2, -3), (3, 3), (-1, 1), (2, -2), (0, 3), (3, 0), (-2, 3).
7. Find the distances between the following pairs of points:

(a) (3, 4) and (3,6) (b) (2, 5) and (2,-2) (c) (3, 1) and (2,1)

(d) (2, 3) and (5, 7) (e) (-2, 4) and (3, 0) (f) $\left(-2, \frac{1}{2}\right)$and (4, -1)

8. Draw the triangle with vertices A(2, 5), B(2, -5), and C(—3, 5), and find its area.
9. If (2, 2), (2, -4), and (5,2) are three vertices of a rectangle, find the fourth vertex.
10. If the points (2, 4) and (-1, 3) are the opposite vertices of a rectangle whose sides are parallel to the coordinate axes (that is, the x and y axes), find the other two vertices.
11. Determine whether the following triples of points are the vertices of an isosceles triangle: (a) (4, 3), (1, 4), (3, 10); (b) (-1, 1), (3, 3), (1, -1); (c) (2, 4), (5, 2), (6, 5).

12. Determine whether the following triples of points are the vertices of a right triangle. For those that are, find the area of the right triangle: (a) (10, 6), (3, 3), (6, -4); (b) (3, 1), (1, -2), (-3, -1); (c) (5, -2), (0, 3),(2,4).
13. Find the perimeter of the triangle with vertices A(4, 9), B(-3, 2), and C(8, -5).
14. Find the value or values of y for which (6, y) is equidistant from (4, 2) and (9, 7).
15. Find the midpoints of the line segments with the following endpoints: (a) (2, -3) and (7, 4); (b) $\left(\frac{5}{3}, 2\right)$ and (4, 1); (c) ($\sqrt{3}$,0) and (1, 4).
16. Find the point (x, y) such that (2, 4) is the midpoint of the line segment connecting (x, y) and (1,5).
17. Determine the point that is equidistant from the points A(-l, 7), B(6, 6), and C(5, -1).